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THE L^p DIRICHLET PROBLEM FOR SMALL PERTURBATIONS OF THE LAPLACIAN

BY

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ABSTRACT

We show that Dahlberg's vanishing trace condition measuring the disagreement between the coefficients of two elliptic operators preserves harmonic measures whose logarithm belongs to VMO.

1. Introduction

In this article we study the weight-regularity properties of the harmonic measure of elliptic operators in non-divergence form on the boundary of a Lipschitz domain D in R^n and whose coefficients are in a certain sense a small perturbation of the Laplacian or other elliptic operator whose harmonic measure is very regular.

According to a result of Dahlberg [D], if the difference between the coefficients of two operators L_0 and L_1 is sufficiently small in an appropriate norm (defined by a Carleson condition) and if the harmonic measure for L_0 lies in the $B_p(d\sigma)$ class, where $1 < p < \infty$ and $d\sigma$ denotes surface measure on the boundary of D, then the same holds for the harmonic measure for L_1 (see the body of the paper for the relevant definitions). In [FKP], the authors proved that if the difference between the coefficients satisfies the same Carleson condition but without the smallness requirement and the harmonic measure for L_0 lies in some $B_p(d\sigma)$ class, then the harmonic measure for L_1 lies in some $B_q(d\sigma)$ class for some q possibly smaller than p. In [F2], the author found other norms measuring the disagreement between the coefficients which guarantee the preservation of the $B_p(d\sigma)$ class but where the corresponding norms are not required to be small.

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In this paper we will prove that if L_0 is an operator such that the logarithm of the density of its harmonic measure with respect to surface measure is a function of Vanishing Mean Oscillation (VMO) on the boundary of D and the disagreement between the coefficients satisfies Dahlberg's Carleson condition with vanishing trace, then the same holds for the harmonic measure for L_1. In particular, if L_0 is the Laplace operator Δ, D is a smooth domain and L_1 is a perturbation of the Laplace operator near the boundary of D satisfying the above condition, then the logarithm of the density of harmonic measure for L_1 is a function of Vanishing Mean Oscillation. This and John–Niremberg's inequality [ST] imply in particular that the harmonic measure for L_1 lies in the Muckenhoupt classes $A_p(d\sigma)$ and $B_p(d\sigma)$ for all $p, 1 < p < \infty$. We recall that for an operator L as above, the Dirichlet problem is solvable for any boundary data in $L_p(d\sigma)$ if and only if its harmonic measure lies in $B_{p'}(d\sigma)$, where $1/p + 1/p' = 1$.

The proof of this theorem follows closely the arguments in [D], but we consider a new type of differential inequality which was first used in [E] to prove that the logarithm of non-negative adjoint solutions to non-divergence form elliptic equations with Vanishing Mean Oscillation coefficients belonging to VMO. Our argument is based on an equivalent condition characterizing those weights whose logarithm lies in VMO and which follows from a theorem (Theorem 3) proved by Sarason [S].

In section 2 we give the precise definitions and state the main theorem, and in section 3 we prove this theorem.

2. Definitions and main theorem

We consider elliptic operators L of the form

$$Lu = \operatorname{div}(A(X)\nabla u)$$

where the coefficient matrix $A(X) = (a_{ij}(X))$ is symmetric and for some $\lambda > 0$

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(X)\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \quad \text{for all } X, \xi \in \mathbb{R}^n. \tag{2.1}$$

We say that a function u in D is a solution to $Lu = 0$ on D provided u is in $W^{1,2}_{\text{loc}}(D)$ and

$$\int_D a_{ij}\partial_j u \partial_i \varphi dx = 0 \quad \text{for all } \varphi \in C_0^\infty(D).$$

For each Q on ∂D, we choose a right circular cone $\gamma(Q)$ with height and aperture fixed, with vertex at Q, and oriented with its axis in the interior normal direction. We define the non-tangential maximal function of a function u in D by

$$N(u)(Q) = \operatorname*{Sup}_{X\in\gamma(Q)} |u(X)| \quad \text{for } Q \text{ on } \partial D.$$

We say that the L^p Dirichlet problem for L on D is solvable if there exists a constant C such that for each $f \in L^p(d\sigma)$ there corresponds a function u in D such that $Lu = 0$ on D,

$$\lim_{\substack{X\in\Gamma(Q)\\ X\to Q}} u(X) = f(Q) \quad \text{for almost every } Q \in \partial D,$$

and

$$\|N(u)\|_{L^p(d\sigma)} \leq C\|f\|_{L^p(d\sigma)}.$$

A measure m on ∂D belongs to $B_p(d\sigma)$,$1 < p < \infty$, when m is mutually absolutely continuous with respect to surface measure σ and there is a constant C satisfying for all Q on ∂D and $r > 0$

$$\left(\frac{1}{\sigma(\Delta_r(Q))}\int_{\Delta_r(Q)} w^p d\sigma\right)^{\frac{1}{p}} \leq C\,\frac{1}{\sigma(\Delta_r(Q))}\int_{\Delta_r(Q)} w\,d\sigma,$$

where $\Delta_r(Q) = \partial D \cap B_r(Q)$ and $w = dm/d\sigma$.

A measure μ on D is called a **Carleson measure** if there is a constant C such that for all Q on ∂D and $r > 0$

$$\mu(\Gamma_r(Q)) \leq C\sigma(\Delta_r(Q)),$$

where $\Gamma_r(Q) = D \cap B_r(Q)$. We say that the Carleson measure μ has **vanishing trace** provided there is a function $h(r)$ with $\lim_{r\to 0} h(r) = 0$ and verifying

$$\mu(\Gamma_r(Q)) \leq h(r)\sigma(\Delta_r(Q))$$

for all Q on ∂D and $r > 0$.

A function β on ∂D has vanishing mean oscillation, $\beta \in \mathrm{VMO}$, provided

$$\lim_{r\to 0}\operatorname*{Sup}_{Q\in\partial D}\left\{\frac{1}{\sigma(\Delta_r(Q))}\int_{\Delta_r(Q)} |\beta - m_{\Delta_r(Q)}(\beta)|d\sigma\right\} = 0,$$

where $m_{\Delta_r(Q)}(\beta)$ denotes the average of β over $\Delta_r(Q)$.

It is well known that for continuous data $f \in C(\partial D)$ there is a unique solution u continuous on $\bar{D}$ to the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{on } D, \\ u = f & \text{on } \partial D. \end{cases}$$

Moreover, there is for each X in D a probability measure ω^X on ∂D satisfying

$$u(X) = \int_{\partial D} f d\omega^X \quad \text{for all } f \in C(\partial D).$$

When the coefficients of L are smooth the harmonic measure for L is always mutually absolutely continuous with respect to surface measure [JK] and the density of harmonic measure for L at X is given by the conormal derivative of the Green function $g(X,Y)$ for L in D [GW]

$$\frac{d\omega^X}{d\sigma}(Q) = \sum_{i,j=1}^{n} a_{ij}(Q)\partial_i g(X,Q)N_j = \frac{\partial g}{\partial \nu}(X,Q),$$

where $N = (N_1, \ldots, N_n)$ is the inner unit normal to ∂D.

We will assume that the domain D is "centered" at the origin 0 of $\mathbb{R}^n$ and denote the harmonic measure for L at 0 as ω; i.e., $\omega = \omega^0$. We recall that the L^p Dirichlet problem is uniquely solvable for L in D if and only if ω lies in $B_{p'}(d\sigma)$ [JK].

We will consider two operators L_0 and L_1 given by

$$L_0 = \operatorname{div}(A_0(X)\nabla u) \quad \text{and} \quad L_1 = \operatorname{div}(A_1(X)\nabla u),$$

where the coefficient matrices $A_i, i = 1,2$ satisfy (2.1) and denote the harmonic measures for these operators as $\omega_i^X, i = 1,2$. We define

$$\begin{aligned} \varepsilon(X) &= (\varepsilon_{ij}(X)) = A_1(X) - A_0(X), \\ a(X) &= \operatorname*{Sup}_{Y \in B(X,\delta(X)/2)} |\varepsilon(Y)|, \end{aligned}$$

where $\delta(X)$ denotes the distance from X to ∂D.

Let us now recall a theorem proved by Dahlberg [D].

THEOREM 1: *Let L_0 and L_1 be two elliptic operators as above, $1 < p < \infty$, D be a Lipschitz domain and suppose that the harmonic measure for L_0 in D belongs to $B_p(d\sigma)$. Then, there exists a constant θ depending on λ, n, p, the $B_p(d\sigma)$ constant of ω_0 and the Lipschitz character of D, such that if for some positive number r_0*

$$\int_{\Gamma_r(Q)} \frac{a(X)^2}{\delta(X)}\, dX \leq \theta\sigma(\Delta_r(Q))$$

for all Q on ∂D and $r \leq r_0$, the harmonic measure for L_1 belongs to $B_p(d\sigma)$, i.e., $\omega_1 \in B_p(d\sigma)$.

In this paper we will prove the following refinement of this theorem.

THEOREM 2: *Let L_0 and L_1 be two elliptic operators as above, D be a Lipschitz domain in $\mathbb{R}^n$ and suppose the measure*

$$d\mu = \frac{a(X)^2}{\delta(X)} dX$$

is a Carleson measure with vanishing trace. Then, if the harmonic measure for L_0 in D is mutually absolutely continuous with respect to surface measure on ∂D and the logarithm of $d\omega_0/d\sigma$ lies in VMO$(d\sigma)$, *the same holds for the harmonic measure of L_1.*

We will say that two objects A and B (numbers or functions) are equivalent and write $A \approx B$ if there exists a positive constant C depending at most on ellipticity λ, dimension n, the Lipschitz character of D and other constants coming from the hypothesis in our theorem such that $C^{-1}A \leq B \leq CA$. Analogously, the notation $A \lesssim B$ will mean that for some C as above, $A \leq CB$.

3. Proof of the theorem

First of all, if m is measure on ∂D mutually absolutely continuous with respect to surface measure such that the logarithm of $dm/d\sigma$ lies in VMO and

$$\varphi(r) = \operatorname*{Sup}_{\substack{Q\in\partial D\\ s\leq r}} \frac{1}{\sigma(\Delta_s(Q))} \int_{\Delta_s(Q)} \left| \log\left(\frac{dm}{d\sigma}\right) - m_{\Delta_s(Q)}\left(\log\left(\frac{dm}{d\sigma}\right)\right)\right| d\sigma,$$

it follows from John–Niremberg's inequality that for Q on ∂D and $s \leq r$

$$\left(\frac{1}{\sigma(\Delta_s(Q))} \int_{\Delta_s(Q)} \left(\frac{dm}{d\sigma}\right)^2 d\sigma\right)^{\frac{1}{2}} \leq (1 + C\varphi(r)) \frac{1}{\sigma(\Delta_s(Q))} \int_{\Delta_s(Q)} \frac{dm}{d\sigma} d\sigma,$$

where C depends on dimension and the Lipschitz character of D. To see this, first observe that $dm/d\sigma$ belongs to $B_p(d\sigma)$ for some $p > 1$ and apply John–Niremberg's inequality to the function $dm/d\sigma = \exp(\log dm/d\sigma)$ with respect to the measure $dm/d\sigma$.

On the other hand, in [S] we find the following inverse to this result.

THEOREM 3: *Let (X, μ) be a measure space, $\mu(X) = 1$. Let $v \geq 0$ and assume that*

$$\int_X v\,d\mu \int_X v^{-1}d\mu \leq 1 + c^3, \quad c < 1/2.$$

Then,

$$\int_X \left| \log v - \log \left[\int_X v\,d\mu \right] \right| d\mu \leq 6c.$$

From this theorem we see that if m is a measure on ∂D verifying $m \in B_2(d\sigma)$ and for some function φ with $\lim_{r\to 0} \varphi(r) = 0$

$$\left(\frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} \left(\frac{dm}{d\sigma}\right)^2 d\sigma \right)^{\frac{1}{2}} \leq (1 + C\varphi(r)) \frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} \frac{dm}{d\sigma} d\sigma$$

for all Q on ∂D and $r > 0$, then the logarithm of $dm/d\sigma$ is a function of Vanishing Mean Oscillation. To see this, we take $X = \Delta_r(Q), v = dm/d\sigma$ and $\mu = m/m(\Delta_r(Q))$ in Theorem 3 obtaining that the logarithm of $dm/d\sigma$ is in VMO with respect to the measure m, and from John–Niremberg's inequality and the fact that m lies in $B_2(d\sigma)$ we obtain that the logarithm of $dm/d\sigma$ lies in VMO with respect to surface measure.

Now, if L_0 and L_1 are as in Theorem 2 we consider for $0 \leq t \leq 1$ the operators

$$L_t u = \operatorname{div}(A_t(X)\nabla u),$$
$$A_t(X) = (1-t)A_0(X) + tA_1(X),$$

and denote the corresponding harmonic measures and Green function for L_t as ω_t, ω_t^X and $g_t(X, Y)$.

From the remark above and the hypothesis in Theorem 2, ω_0 is a $B_2(d\sigma)$-weight, and from Dahlberg's theorem the same holds for ω_t for $0 \leq t \leq 1$ and with a uniform $B_2(d\sigma)$-constant.

To simplify our exposition we will assume that D coincides with the unit ball B in $\mathbb{R}^n$ and consider for each $0 \le t \le 1$ the solution v_t to

$$\begin{cases} L_t v_t = -\varphi & \text{on } B \\ v_t = 0 & \text{on } \partial B \end{cases}$$

where $\varphi \in C_0^\infty(B)$ with $\varphi = 1$ on $B_{1/4}$ and $\varphi = 0$ outside of $B_{1/2}$, that is

$$v_t(X) = \int_B g_t(Y, X)\varphi(Y)dY.$$

From Harnak's inequality [M], the comparison principle for non-negative solutions to non-divergence form equations and estimates for the Green function [CFMS] we have

$$0 \le v_t(X) \lesssim 1,$$

$$\frac{\partial g_t}{\partial \nu}(0, Q) = \frac{d\omega_t}{d\sigma} \approx \frac{\partial v_t}{\partial \nu}(Q) \quad \text{a.e. on } \partial B \tag{3.1}$$

and

$$v_t(X) \approx g_t(0, X) \approx \frac{\omega\left(\Delta_{\delta(X)}\left(\frac{X}{|X|}\right)\right)}{\delta(X)^{n-2}} \quad \text{for } |X| > \frac{1}{2}, \tag{3.2}$$

where $\delta(X)$ denotes the distance from X to ∂B. On the other hand, the ratio

$$\frac{\partial g_t}{\partial \nu}(0, Q) \Big/ \frac{\partial v_t}{\partial \nu}(Q)$$

defines a Hölder continuous function on ∂B and with constants depending only on λ and n [JK]. Thus, the logarithm of $d\omega_t/d\sigma$ lies in VMO if and only if the same holds for the logarithm of the weight function $k_t = \partial v_t/\partial \nu$.

Using these results we will prove the following lemma.

LEMMA: *Let L_0 and L_1 be as in Theorem 2. Then, there exists a function Φ with $\lim_{r\to 0}\Phi(r) = 0$ verifying*

$$\left(\frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} k_1^2 d\sigma\right)^{\frac{1}{2}} \le (1 + \Phi(r)) \frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} k_1 d\sigma$$

for all $Q \in \partial D$ and $r > 0$.

From this lemma and the previous remarks our theorem follows.

To prove this lemma we write for E contained on ∂B and $0 \le t \le 1$

$$k_t(E) = \int_E k_t d\sigma$$

and we need to estimate for fixed Q_0 on ∂D and $r > 0$

$$\frac{1}{k_1(\Delta_r(Q_0))} \times \operatorname{Sup} \left\{ \int_{\Delta_r(Q_0)} f k_1 d\sigma \colon \operatorname{Supp}(f) \subset \Delta_r(Q_0), f \ge 0, \|f\|_{L^2(d\sigma/\sigma(\Delta_r(Q_0)))} \le 1 \right\}.$$

Fixing $Q_0, r > 0$ and f as above we write $\Delta_s = \Delta_s(Q_0)$ and $\Gamma_s = \Gamma_s(Q_0)$ for $s > 0$, and consider on $[0,1]$ the function

$$\Psi(t) = \frac{1}{k_t(\Delta_r)} \int_{\Delta_r} f k_t d\sigma.$$

Using standard arguments it is simple to show that for fixed $f \in L^2(d\sigma)$ the function Ψ is Lipschitz on $[0,1]$ and its derivative is given by

$$\dot{\Psi}(t) = \frac{1}{k_t(\Delta_r)} \int_{\Delta_r} \dot{k}_t [f - m_{\Delta_r, k_t}(f)] d\sigma,$$

where $m_{\Delta_r,k_t}(f)$ denotes the average of f over Δ_r with respect to k_t, $\dot{k}_t$ belongs to $L^2(d\sigma)$ with $\|\dot{k}_t\|_{L^2(d\sigma)} \lesssim 1$ for $0 \le t \le 1$ and $\dot{k}_t$ is the weak limit in $L^2(d\sigma)$ of $(k_{t+h} - k_t)/h$ as h tends to zero. Moreover, since $\varepsilon(Q)$ is the zero matrix on ∂B

$$\dot{k}_t = \sum_{i,j=1}^{n} a_{ij}(Q) \partial_i \dot{v}_t(Q) N_j = \frac{\partial \dot{v}_t}{\partial \nu}(Q) \quad \text{a.e. on } \partial B,$$

where $\dot{v}_t$ is the weak limit in $W_0^{1,2}(B)$ of $(v_{t+h} - v_t)/h$ as h tends to zero and where $\dot{v}_t$ satisfies for X in B

$$\begin{gathered} \operatorname{div}(A_t(X)\nabla \dot{v}_t(X)) + \operatorname{div}(\varepsilon(X)\nabla v_t(X)) = 0, \\ \dot{v}_t = 0 \quad \text{on } \partial D. \end{gathered}$$

We will show that there are positive numbers γ, β and $C, 0 < \beta,\ \gamma < 1$ depending on λ, n and the $B_2(d\sigma)$ constant of ω_0 verifying for $0 \le t \le 1$

$$|\dot{\Psi}(t)| \le C\left[r^\gamma + \operatorname*{Sup}_{\substack{Q \in \partial B \\ s \le r^\beta}} \left(\frac{1}{\sigma(\Delta_s(Q))} \int_{\Gamma_s(Q)} \frac{a(X)^2}{\delta(X)} dX \right)^{\frac{1}{2}} \right]. \tag{3.3}$$

From this estimate and the fundamental theorem of calculus we obtain

$$\frac{1}{k_1(\Delta_r)}\int_{\Delta_r} fk_1 d\sigma \le 1+\Phi_0 + C\left[r^\gamma + \operatorname*{Sup}_{\substack{Q\in\partial B\\ s\le r^\beta}}\left(\frac{1}{\sigma(\Delta_s(Q))}\int_{\Gamma_s(Q)}\frac{a(X)^2}{\delta(X)}dX\right)^{\frac{1}{2}}\right],$$

where Φ_0 is the function with $\lim_{r\to 0}\Phi_0(r)=0$ associated to the Vanishing Mean Oscillation condition of the logarithm of k_0. This estimate and duality imply the lemma above.

To prove (3.3) we consider the solution u_t to

$$\begin{cases} L_t u_t = & \text{on } B\\ u_t = h_t & \text{on } \partial B\end{cases}$$

where

$$h_t = \frac{1}{k_t(\Delta_r)}(f - m_{\Delta_r,k_t}(f))\chi_{\Delta_r}.$$

We have for X in B

$$u_t(X) = \int_{\partial B} h_t(Q)d\omega_t^X = \int_{\partial B} G_t(X,Q)h_t(Q)k_t(Q)d\sigma,$$

where $G_t(X,Q)$ for X in B and Q on ∂B denotes the function

$$G_t(X,Q) = \lim_{s\to 0}\frac{g_t(X,(1-s)Q)}{v_t((1-s)Q)}.$$

Observing that h_t has zero average with respect to k_t, we can write

$$u_t(X) = \int_{\partial B}[G_t(X,Q)-G_t(X,Q_0)]h_t(Q)k_t(Q)d\sigma.$$

On the other hand, it is known that $G_t(X,.)$ is α-Hölder continuous on ∂B for some α depending on λ and n [JK], and for X in B and Q on the boundary of B with $|Q-Q_0|\le \frac{1}{2}|X-Q_0|$

$$|G_t(X,Q)-G_t(X,Q_0)| \lesssim \left(\frac{|Q-Q_0|}{|X-Q_0|}\right)^\alpha \frac{1}{\omega_t(\Delta_{|X-Q_0|})}.$$

Hence, from the support properties of h_t, the above estimate and Schwartz's inequality we have for $|X-Q_0|\ge 2r$

$$|u_t(X)| \lesssim \left(\frac{r}{|X-Q_0|}\right)^\alpha \frac{1}{\omega_t(\Delta_{|X-Q_0|})}\frac{\sigma(\Delta_r)}{k_t(\Delta_r)}\left(\frac{1}{\sigma(\Delta_r)}\int_{\Delta_r} k_t^2 d\sigma\right)^{\frac{1}{2}},$$

and from Dahlberg's theorem we know that k_t lies in $B_2(d\sigma)$ with a uniform constant. Thus, for $|X - Q_0| \geq 2r$

$$|u_t(X)| \lesssim \left(\frac{r}{|X - Q_0|}\right)^{\alpha} \frac{1}{\omega_t(\Delta_{|X-Q_0|})}. \tag{3.4}$$

Now, integration by parts together with the boundary values of v_t, $\dot{v}_t$ and $\epsilon(X)$ yield

$$\dot{\Psi}(t) = \int_B \varepsilon_{ij}\partial_j v_t \partial_i u_t dX,$$

$$|\dot{\Psi}(t)| \lesssim \int_B |\varepsilon(X)||\nabla v_t||\nabla u_t| dX.$$

To estimate this integral we let β denote a small positive number to be chosen later and write

(3.5)

$$\int_B |\varepsilon(X)||\nabla v_t||\nabla u_t|dX = \int_{B\backslash\Gamma_{r^\beta}} |\varepsilon(X)||\nabla v_t||\nabla u_t|dX + \int_{\Gamma_{r^\beta}} |\varepsilon(X)||\nabla v_t||\nabla u_t|dX.$$

From Schwartz's inequality

$$\int_{B\backslash\Gamma_{r^\beta}} |\varepsilon(X)||\nabla v_t||\nabla u_t|dX \lesssim \left(\int_B |\nabla v_t|^2 dX\right)^{\frac{1}{2}} \left(\int_{B\backslash\Gamma_{r^\beta}} |\nabla u_t|^2 dX\right)^{\frac{1}{2}}. \tag{3.6}$$

Since $u_t = 0$ on $\partial B\backslash\Delta_r$, we have from Cacciopoli's inequality

$$\begin{aligned}\left(\int_{B\backslash\Gamma_{r^\beta}} |\nabla u_t|^2 dX\right)^{\frac{1}{2}} &\lesssim r^{-\beta}\left(\int_{\Gamma_{2r^\beta}\backslash\Gamma_{r^\beta/2}} u_t^2 dX\right)^{\frac{1}{2}} \\ &\lesssim r^{-\beta(1-n/2)}\text{Sup}\{|u_t(X)|: X \in \Gamma_{2r^\beta}: \Gamma_{r^\beta/2}\}\end{aligned}$$

and from (3.4) and the doubling property of harmonic measure [CFMS]

$$\text{Sup}\{|u_t(X)|: X \in \Gamma_{2r^\beta}\backslash\Gamma_{r^\beta/2}\} \lesssim r^{(1-\beta)\alpha}\frac{1}{\omega_t(\Delta_{r^\beta})}.$$

Hence,

$$\left(\int_{B\backslash\Gamma_{r^\beta}} |\nabla u_t|^2 dX\right)^{\frac{1}{2}} \lesssim r^{-\beta(1-n/2)+(1-\beta)\alpha}\,\frac{1}{\omega_t(\Delta_{r^\beta})}. \tag{3.7}$$

On the other hand, integration by parts and Poincaré's inequality imply

$$\int_B |\nabla v_t|^2 dX \lesssim 1, \tag{3.8}$$

and from (3.6), (3.7) and (3.8) we obtain

$$\int_{B\setminus\Gamma_{r^\beta}} |\varepsilon(X)||\nabla v_t||\nabla u_t| dX \lesssim r^{-\beta(1-n/2)+(1-\beta)\alpha} \frac{1}{\omega_t(\Delta_{r^\beta})}.$$

Recalling that ω_t belongs to $B_2(d\sigma)$ with a uniform constant on t, we have for some constant $\rho > 0$ depending on λ, n, and the $B_2(d\sigma)$ constant of ω_0, $s^\rho \lesssim \omega_t(\Delta_s)$ for all $0 < s < 1$ [ST], implying

$$\int_{B\setminus\Gamma_{r^\beta}} |\varepsilon(X)||\nabla v_t||\nabla u_t| dX \lesssim r^{\alpha-\beta(1-n/2+\alpha+\rho)} = r^\gamma,$$

where γ will be positive after choosing the number β sufficiently small.

To estimate the second integral on the right hand side of (3.5) we fix $N \geq 4$ with $2^{N+1}r \approx r^\beta$ and we have

$$\begin{aligned}\int_{\Gamma_{r^\beta}} |\varepsilon(X)||\nabla v_t||\nabla u_t| dX &\leq I + \sum_{j=3}^{N} \Pi_j \\ &= \int_{\Gamma_{8r}} |\varepsilon(X)||\nabla v_t||\nabla u_t| dX \\ &\quad + \sum_{j=3}^{N} \int_{\Gamma_{2^{j+1}r}\setminus\Gamma_{2^j r}} |\varepsilon(X)||\nabla v_t||\nabla u_t| dX.\end{aligned}$$

Letting Ω denote a dyadic decomposition of Δ_{8r} and defining for each dyadic surface cap J in Ω

$$\Gamma(J) = \{X \in B\colon \frac{X}{|X|} \in J, \ell(J) \leq \delta(X) \leq 2\ell J)\}, \quad \ell(J) = \sigma(J)^{1/(n-1)},$$

we have from (3.2), Cacciopoli's inequality and the doubling property of harmonic measure

$$\begin{aligned} I &\lesssim \sum_{J\in\Omega} \Big(\int_{\Gamma(J)} \frac{a(X)^2}{\delta(X)^n} H_t(X)^2 dX \Big)^{\frac12} \Big(\int_{\Gamma(J)} \delta(X)^{2-n} |\nabla u_t|^2 dX \Big)^{\frac12} \sigma(J) \\ &= \sum_{J\in\Omega} a_j b_j \sigma(J),\end{aligned}$$

where

$$H_t(X) = \frac{\omega_t\left(\Delta_{\delta(X)}\left(\frac{X}{|X|}\right)\right)}{\delta(X)^{n-1}} \quad \text{for } X \text{ in } \Gamma_{8r}$$

and identically zero otherwise. We define functions F and G from Δ_{8r} into $\ell^2(\Omega)$ as $F(Q) = (\chi_J(Q)a_J)_{J\in\Omega}$ and $G(Q) = (\chi_J(Q)b_j)_{J\in\Omega}$ for Q in Δ_{8r}. We have

$$\|F(Q)\|_{\ell^2(\Omega)} \lesssim \Bigg(\int\limits_{\gamma(Q)} \frac{a(X)^2}{\delta(X)^n} H_t(X)^2 dX \Bigg)^{\frac{1}{2}}, \quad \|G(Q)\|_{\ell^2(\Omega)} \lesssim S(u_t)(Q),$$

where $S(u_t)$ denotes the Lusin Area function of u_t. Then

$$\sum_{J\in\Omega} a_J b_J \sigma(J) \lesssim \int\limits_{\Delta_{8r}} F(Q)\cdot G(Q) d\sigma$$

and from Schwartz's inequality and Fubini, the right-hand side above is bounded by

$$\Bigg(\int\limits_{\partial B} \frac{a(X)^2}{\delta(X)} H_t(X)^2 dX \Bigg)^{\frac{1}{2}} \Bigg(\int\limits_{\partial B} S(u_t)^2 d\sigma \Bigg)^{\frac{1}{2}}.$$

Since ω_t lies in $B_2(d\sigma)$ with a uniform constant on $t \in [0,1]$, the L^2 norm of the Lusin Area integral of u_t is bounded uniformly by the L^2 norm of its boundary values h_t [DJK], and

$$\|h_t\|_{L^2(d\sigma)} \lesssim \frac{\sigma(\Delta_r)^{1/2}}{\omega_t(\Delta_r)}.$$

The non-tangential maximal function of H_t is bounded by the Hardy–Littlewood maximal function with respect to surface measure of $(d\omega_t/d\sigma)\chi_{\Delta_{10r}}$, which has L^2 norm bounded by $\omega_t(\Delta_r)/\sigma(\Delta_r)^{1/2}$. From these remarks and a standard Carleson measure argument

$$\Bigg(\int\limits_{\partial B} \frac{a(X)^2}{\delta(X)} H_t(X)^2 dX \Bigg)^{\frac{1}{2}} \lesssim \frac{\omega_t(\Delta_r)}{\sigma(\Delta_r)^{1/2}} \operatorname*{Sup}_{\substack{Q\in\partial B \\ s\leq r^\beta}} \Bigg(\frac{1}{\sigma(\Delta_s(Q))} \int\limits_{\Gamma_s(Q)} \frac{a(X)^2}{\delta(X)} dX \Bigg)^{\frac{1}{2}}.$$

Thus,

$$I \lesssim \operatorname*{Sup}_{\substack{Q\in\partial B \\ s\leq r^\beta}} \Bigg(\frac{1}{\sigma(\Delta_s(Q))} \int\limits_{\Gamma_s(Q)} \frac{a(X)^2}{\delta(X)} dX \Bigg)^{\frac{1}{2}}. \tag{3.9}$$

To estimate Π_j for $j \geq 3$ we let Ω_j denote a dyadic decomposition of $\Delta_{2^{j+1}r} \backslash \Delta_{2^j r}$ and observe that $(\Gamma_{2^{j+1}r} \backslash \Gamma_{2^j r}) \backslash \bigcup_{j\in\Omega_j} \Gamma(J)$ is essentially contained in a ball centered at the point $(1-2^j r)Q_0$ of radius $2^{j-1}r$. From this remark, Cacciopoli's inequality, (3.2), (3.4) and the doubling property of harmonic measure we have

$$\begin{aligned}\Pi_j &\lesssim 2^{-\alpha j} \operatorname*{Sup}_{\substack{Q\in\partial B\\ s\le r^\beta}} \left(\frac{1}{\sigma(\Delta_s(Q))}\int_{\Gamma_s(Q)} \frac{a(X)^2}{\delta(X)}dX\right)^{\frac12} \\ &\quad + \sum_{J\in\Omega_j} \ell(J)^{3/2-n}\left(\int_{\Gamma(J)} \frac{a(X)^2}{\delta(X)}dX\right)^{\frac12}\left(\int_{\Gamma(J)}|\nabla u_t|^2 dX\right)^{\frac12}\omega_t(J).\end{aligned} \tag{3.10}$$

We observe that for $J\in\Omega_j$, $\Gamma(J)$ is contained in $\Gamma_{2^{j+2}r}\backslash\Gamma_{2^{j-1}r}$ and the boundary values of u_t are supported in Δ_r. Hence, from standard estimates for solutions to $L_t u=0$ vanishing on a boundary portion and (3.4) we have for X in $J\in\Omega_j$

$$|u_t(X)| \lesssim \left(\frac{\ell(J)}{2^j r}\right)^\alpha \operatorname*{Sup}_{\Gamma_{2^{j+2}r}\backslash\Gamma_{2^{j-1}r}} |u_t| \lesssim \left(\frac{\ell(J)}{2^j r}\right)^\alpha \frac{2^{-\alpha j}}{\omega_t(\Delta_{2^j r})}.$$

From Cacciopoli's inequality and the above inequality, (3.10) is bounded by

$$\begin{aligned}&\frac{2^{-\alpha j}}{\omega_t(\Delta_{2^j r})}\sum_{J\in\Omega_j}\left(\int_{\Gamma(J)}\frac{a(X)^2}{\delta(X)}dX\right)^{\frac12}\frac{\omega_t(J)}{\sigma(J)^{1/2}}\left(\frac{\ell(J)}{2^j r}\right)^\alpha \\ &\lesssim \frac{2^{-\alpha j}}{\omega_t(\Delta_{2^j r})}\operatorname*{Sup}_{\substack{Q\in\partial B\\ s\le r^\beta}}\left(\frac{1}{\sigma(\Delta_s(Q))}\int_{\Gamma_s(Q)}\frac{a(X)^2}{\delta(X)}dX\right)^{\frac12} \\ &\quad\times\sum_{i=0}^{\infty}\sum_{J\in\Omega_j,\ell(J)\approx 2^{j-i}r}\omega_t(J)2^{-\alpha i} \\ &\lesssim 2^{-\alpha j}\operatorname*{Sup}_{\substack{Q\in\partial B\\ s\le r^\beta}}\left(\frac{1}{\sigma(\Delta_s(Q))}\int_{\Gamma_s(Q)}\frac{a(X)^2}{\delta(X)}dX\right)^{\frac12}.\end{aligned}$$

Hence,

$$\Pi_j \lesssim 2^{-\alpha j}\operatorname*{Sup}_{\substack{Q\in\partial B\\ s\le r^\beta}}\left(\frac{1}{\sigma(\Delta_s)(Q))}\int_{\Gamma_s(Q)}\frac{a(X)^2}{\delta(X)}dX\right)^{\frac12}. \tag{3.11}$$

Therefore, from (3.9) and (3.11)

$$\int_{\Gamma_{r\beta}}|\varepsilon(X)||\nabla v_t||\nabla u_t|dX \lesssim I+\sum_{j=3}^{N}\Pi_j \lesssim \operatorname*{Sup}_{\substack{Q\in\partial B\\ s\le r^\beta}}\left(\frac{1}{\sigma(\Delta_s(Q))}\int_{\Gamma_s(Q)}\frac{a(X)^2}{\delta(X)}dX\right)^{\frac12}$$

proving the lemma.

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